Triangular lattice points close to the origin

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Introduction

The number of triangular lattice points within a circle of integer radius centred on a lattice point is

$$f(r) = 1 + 6 \sum_{k=0}^{\infty} \left\lfloor \frac{r^2}{3k+1} \right\rfloor - \left\lfloor \frac{r^2}{3k+2} \right\rfloor$$

The purpose of this note is to prove this while requiring little background knowledge beyond basic properties of complex numbers, the concept of a generating function[1], and some very elementary differential calculus ($\frac{\mathrm{d}}{\mathrm{d}x} x^a = a x^{a-1}$ and the chain rule).

We can identify the lattice points with the Eisenstein integers, numbers of the form $x + y\omega$ where $x, y \in \mathbb{Z}$ and ω is a primitive cube root of unity. For convenience, however, we shall work with a different parameterisation: $x - y\omega$. Clearly the lattice points themselves are the same and can be placed into bijection. Therefore we seek to count $x, y \in \mathbb{N}$ for which $|x - y\omega| \leq r$, or $|x - y\omega|^2 \leq r^2$. As with any complex number, we find this norm by multiplying with the conjugate: $|x - y\omega|^2 = (x - y\omega)(x - y\omega)^* = x^2 + xy + y^2$.

The approach taken is to count $x, y \in \mathbb{N}$ for which $x^2 + xy + y^2 = k$ and then sum over $0 \leq k \leq r^2$. The key is the generating function identity

$$\sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2} = 1 + 6 \sum_{n \geq 0} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right)$$

and its proof follows Hirschhorn [3]. A key element of the proof is an identity due to Jacobi, which follows from the so-called Jacobi triple product identity. The latter identity is therefore where we shall start.

[1] In fact, functions will be treated throughout as generating functions of purely combinatoric interest, so we will completely ignore questions of convergence which the reader who is familiar with real or complex analysis might raise.

Jacobi triple product identity

We use some notation from q-series. Let $(a;q)_n$ denote $\prod_{k=0}^{n}(1-aq^k)$ with shorthand notation $(q)_n = (q;q)_n = \prod_{k=1}^{n}(1-q^k)$.

Lemma 1. $(1-a)(aq;q)_n = (a;q)_{n+1} = (1-aq^n)(a;q)$

Proof: the first equality is by reindexing and absorbing the extra term; the second simply extracts the largest term from the product. Note that in the infinite case we only use the first equality: $(1-a)(aq;q)_\infty = (a;q)_\infty$.

Theorem 1 (Euler)**.** $(-x;q)_\infty = \sum_{k=0}^{\infty} q^{k(k-1)/2}(q)_k^{-1}x^k$

The proof is inspired by a proof of a related theorem[4]: $(-x;q)_\infty$ is a product of terms which are all linear in x; therefore we expand it as

$$(-x;q)_\infty = \sum_{k=0}^{\infty} Q_k(q)x^k$$

Then by lemma 1,

$$\sum_{k=0}^{\infty} Q_k(q)x^k = (1+x)\sum_{k=0}^{\infty} Q_k(q)(xq)^k$$

so that equating coefficients in x^k,

$$Q_k(q) = q^k Q_k(q) + q^{k-1}Q_{k-1}(q)$$

or

$$Q_k(q) = q^{k-1}(1-q^k)^{-1}Q_{k-1}(q)$$

Q_0 is the product of the part of each term which is independent of x, and that part is 1, so $Q_0 = 1$. Then by induction,

$$Q_k(q) = q^{\sum_{j=1}^{k}(j-1)}\prod_{j=1}^{k}(1-q^j)^{-1}Q_0 = q^{k(k-1)/2}(q)_k^{-1}$$

as desired.[2] □

[2] A similiar approach constitutes the first half of the proof in Hardy and Wright [2, theorem 352] of theorem 3: denoting $(q;q)_\infty(z;q)_\infty(z^{-1}q;q)_\infty$ as $P(z,q)$, we have

$$\begin{aligned} P(zq,q) &= (q;q)_\infty(zq;q)_\infty(z^{-1};q)_\infty \\ &= (q;q)_\infty(z;q)_\infty(1-z)^{-1}(1-z^{-1})(z^{-1}q;q)_\infty \\ &= -z^{-1}P(z,q) \end{aligned}$$

If we now express $P(z,q) = \sum_{n=-\infty}^{\infty} a_n(q)z^n$ then by equating coefficients of z^n in $P(z,q)$ and $-zP(zq,q)$ we find that $a_n(q) = -q^{n-1}a_{n-1}(q)$ from which, by induction, $a_n(q) = (-1)^n q^{n(n-1)/2}a_0(q)$. However, to finish the proof by showing that $a_0(q) = 1$ requires going into analytical considerations which are outside the scope of this note.

Theorem 2 (Euler)**.** $(-x;q)_\infty^{-1} = \sum_{k=0}^\infty (-1)^k (q)_k^{-1} x^k$

Proof procedes similarly. Since $(1+xq^k)^{-1} = \sum_{j=0}^\infty (-xq^k)^j$ (geometric progression), we again argue that we can expand as a polynomial in x:

$$(-x;q)_\infty^{-1} = \sum_{k=0}^\infty R_k(q) x^k$$

Then substituting into lemma 1 and equating coefficients of x^k we have

$$q^k R_k(q) = R_k(q) + R_{k-1}(q)$$

so

$$R_k(q) = -(1-q^k)^{-1} R_{k-1}(q)$$

$R_0 = 1$ either by considering the constant term in the geometric progression or by arguing that it must be the reciprocal of the constant term in the RHS of theorem 1 since $(q;q)_\infty (q;q)_\infty^{-1} = 1$; and by induction $R_k(q) = (-1)^k (q)_k^{-1}$ □

Theorem 3 (Jacobi)**.** $(q)_\infty(-xq;q)_\infty(-x^{-1};q)_\infty = \sum_{k\in\mathbb{Z}} q^{k(k+1)/2} x^k$

The proof follows the lines of Andrews'[1]. Using theorem 1,

$$(-xq;q)_\infty = \sum_{k=0}^\infty q^{k(k+1)/2} (q)_k^{-1} x^k$$

Now,

$$(q)_n^{-1} = \frac{1}{\prod_{k=1}^n (1-q^k)} = \frac{\prod_{k=n+1}^\infty (1-q^k)}{\prod_{k=1}^\infty (1-q^k)} = \frac{(q^{n+1};q)_\infty}{(q)_\infty}$$

so

$$\begin{aligned}
(-xq;q)_\infty &= \sum_{k=0}^\infty q^{k(k+1)/2} \frac{(q^{k+1};q)_\infty}{(q)_\infty} x^k \\
(q)_\infty(-xq;q)_\infty &= \sum_{k=0}^\infty q^{k(k+1)/2} (q^{k+1};q)_\infty x^k \\
&= \sum_{k=-\infty}^\infty q^{k(k+1)/2} (q^{k+1};q)_\infty x^k
\end{aligned}$$

since $(q^{k+1};q)_\infty$ includes a multiplicative term $(1-q^0)$ when $k<0$. Now, expanding it with theorem 1 we get

$$\begin{aligned}
(q)_\infty(-xq;q)_\infty &= \sum_{k=-\infty}^{\infty} q^{k(k+1)/2}x^k \sum_{m=0}^{\infty} q^{m(m-1)/2}(q)_m^{-1}(-q^{k+1})^m \\
&= \sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty} (-1)^m q^{k(k+1)/2+m(m-1)/2+(k+1)m}(q)_m^{-1}x^k \\
&= \sum_{m=0}^{\infty}(-1)^m(q)_m^{-1}x^{-m} \sum_{k=-\infty}^{\infty} q^{(k+m)(k+m+1)/2}x^{k+m} \\
&= \left(\sum_{m=0}^{\infty}(-1)^m(q)_m^{-1}x^{-m}\right)\left(\sum_{k=-\infty}^{\infty} q^{k(k+1)/2}x^k\right)
\end{aligned}$$

where the last step is by reindexing the sum over k. Then by theorem 2,

$$(q)_\infty(-xq;q)_\infty = (-x^{-1};q)_\infty^{-1} \sum_{k=-\infty}^{\infty} q^{k(k+1)/2}x^k$$

□

A different triple product identity due to Jacobi

Theorem 4 (Jacobi)**.** $(q)_\infty^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}$

Here we can follow Hardy and Wright [2]. We start by applying lemma 1 to $(-x^{-1};q)_\infty$ in theorem 3 to yield

$$(q)_\infty(-xq;q)_\infty(-x^{-1}q;q)_\infty = (1+x^{-1})^{-1}\sum_{k\in\mathbb{Z}} q^{k(k+1)/2}x^k$$

Since $k(k+1) = (-k-1)(-k)$ we can pair up the terms on the RHS to give

$$\begin{aligned}
(q)_\infty(-xq;q)_\infty(-x^{-1}q;q)_\infty &= (1+x^{-1})^{-1}\sum_{k=0}^{\infty} q^{k(k+1)/2}(x^k + x^{-k-1}) \\
&= \sum_{k=0}^{\infty} q^{k(k+1)/2}\left(\frac{1+x^{2k+1}}{1+x}\right)x^{-k} \\
&= \sum_{k=0}^{\infty} q^{k(k+1)/2}(x^{2n} - x^{2n-1} \pm \cdots - x + 1)x^{-k}
\end{aligned}$$

Substituting $x = -1$ we obtain the desired result. □

Counting Eisenstein integers with a given norm

This is a fairly straight exposition of Hirschhorn [3].

Theorem 5. $\sum_{m,n\in\mathbb{Z}} \omega^{m-n} q^{m^2+mn+n^2} = (q)_\infty^3 (q^3)_\infty^{-1}$

Recall that ω is a primitive cube root of unity. We'll introduce a bit of notation: $[x^j]\sum_k a_k x^k = a_j$ is the coefficient extraction operator, which we've already used implicitly when we identified coefficients in x^k in earlier proofs.

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} \omega^{m-n} q^{m^2+mn+n^2} &= \sum_{m+n+p=0} \omega^{m-n} q^{(m^2+n^2+p^2)/2} \\ &= [x^0]\left(\sum_{m=-\infty}^{\infty} \omega^m q^{m^2/2} x^m \sum_{n=-\infty}^{\infty} \omega^{-n} q^{n^2/2} x^n \sum_{p=-\infty}^{\infty} q^{p^2/2} x^p\right)\end{aligned}$$

Let's take a representative one of the three sums in that product: $\sum_{m=-\infty}^{\infty} \omega^m q^{m^2/2} x^m = \sum_{m=-\infty}^{\infty} q^{m(m+1)/2} (\omega x q^{-1/2})^m = (q)_\infty(-\omega x\sqrt{q};q)_\infty(-\frac{\sqrt{q}}{\omega x};q)_\infty$ by theorem 3.
So

$$\text{LHS} = [x^0](q)_\infty^3(-\omega x\sqrt{q};q)_\infty(-\frac{\sqrt{q}}{\omega x};q)_\infty(-\frac{x\sqrt{q}}{\omega};q)_\infty(-\frac{\omega\sqrt{q}}{x};q)_\infty(-x\sqrt{q};q)_\infty(-\frac{\sqrt{q}}{x};q)_\infty$$

Now, consider that by parts.

$$\begin{aligned}(-\omega x\sqrt{q};q)_\infty(-\frac{x\sqrt{q}}{\omega};q)_\infty(-x\sqrt{q};q)_\infty &= \prod_{k=0}^{\infty}(1+\omega x\sqrt{q}q^k)(1+\frac{x\sqrt{q}}{\omega}q^k)(1+x\sqrt{q}q^k) \\ &= \prod_{k=0}^{\infty}(1+x^3q^{3k+3/2}) \\ &= (-x^3q^{3/2};q^3)_\infty\end{aligned}$$

Similarly

$$(-\frac{\sqrt{q}}{\omega x};q)_\infty(-\frac{\omega\sqrt{q}}{x};q)_\infty(-\frac{\sqrt{q}}{x};q)_\infty = (-x^{-3}q^{3/2};q^3)_\infty$$

So

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} \omega^{m-n} q^{m^2+mn+n^2} &= [x^0](q)_\infty^3(-x^3q^{3/2};q^3)_\infty(-x^{-3}q^{3/2};q^3)_\infty \\ &= \frac{(q)_\infty^3}{(q^3)_\infty}[x^0](q^3)_\infty(-x^3q^{3/2};q^3)_\infty(-x^{-3}q^{3/2};q^3)_\infty \\ &= \frac{(q)_\infty^3}{(q^3)_\infty}[x^0]\sum_{k\in\mathbb{Z}} q^{3k(k+1)/2}(x^3q^{-3/2})^k \\ &= \frac{(q)_\infty^3}{(q^3)_\infty}\end{aligned}$$

□

Theorem 6 (Lorenz)**.**

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2} = 1 + 6\sum_{k\geq 0}\left(\frac{q^{3k+1}}{1-q^{3k+1}} - \frac{q^{3k+2}}{1-q^{3k+2}}\right)$$

We start with the left hand side of theorem 5. We can split the sum into three according to the value of ω^{m-n} and write

$$\sum_{m-n\equiv 0 \bmod 3} q^{m^2+mn+n^2} = \sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^2}$$

using the substitution $m = 2k + l$, $n = l - k$; and

$$\sum_{m-n\equiv 1 \bmod 3} q^{m^2+mn+n^2} = \sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^2+3k+3l+1}$$

using the substitution $m = 2k + l + 1$, $n = l - k$. Since m and n are symmetric in $m^2 + mn + n^2$ the sum for the case $m - n \equiv -1 \bmod 3$ is identical. $\omega + \omega^{-1} = -1$, so

$$\sum_{m,n\in\mathbb{Z}} \omega^{m-n} q^{m^2+mn+n^2} = \sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^2} - \sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^2+3k+3l+1}$$

Note that in one of those sums the powers of q are equal to $0 \pmod 3$ and in the other sum they're equal to $1 \pmod 3$. If we now look at the right hand side of theorem 5, $(q^3)_\infty^{-1}$ expands to a polynomial in q^3. So substituting theorem 4 into theorem 5 and equating coefficients of powers of q equal to $0 \pmod 3$ we have

$$\begin{aligned}
(q^3)_\infty \sum_{k,l\in\mathbb{Z}} q^{3k^2+3kl+3l^2} &= \sum_{k\geq 0, k(k+1)/2\equiv 0 \bmod 3} (-1)^k (2k+1) q^{k(k+1)/2} \\
&= \sum_{k\geq 0, k\not\equiv 1 \bmod 3} (-1)^k (2k+1) q^{k(k+1)/2} \\
&= \sum_{n=0}^{\infty} (-1)^{3n}(6n+1) q^{3n(3n+1)/2} + \sum_{n=1}^{\infty} (-1)^{3n-1}(6n-1) q^{(3n-1)(3n)/2} \\
&= \sum_{n=0}^{\infty} (-1)^{3n}(6n+1) q^{3n(3n+1)/2} + \sum_{n=-\infty}^{-1} (-1)^{3n}(6n+1) q^{(3n+1)(3n)/2}
\end{aligned}$$

where in the last step we substitute $-n$ for n in the second sum. Now if we combine the sums and substitute $q^{1/3}$ for q,

$$(q)_\infty \sum_{k,l\in\mathbb{Z}} q^{k^2+kl+l^2} = \sum_{n=-\infty}^{\infty} (-1)^n (6n+1) q^{n(3n+1)/2}$$

To finish it off we need a bit of calculus, and in particular the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\prod_i f_i(x) = \sum_j \frac{\prod_i f_i(x)}{f_j(x)}\frac{\mathrm{d}}{\mathrm{d}x}f_j(x) = \left(\prod_i f_i(x)\right)\sum_j \frac{1}{f_j(x)}\frac{\mathrm{d}}{\mathrm{d}x}f_j(x)$$

We introduce a suitable dummy variable into the right hand side of our most recent sum:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)x^{6n}q^{n(3n+1)/2} &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=-\infty}^{\infty}(-1)^n x^{6n+1}q^{n(3n+1)/2} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\sum_{n=-\infty}^{\infty}q^{3n(n+1)/2}(-q^{-1}x^6)^n\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x(q^3)_\infty(x^6q^2;q^3)_\infty(x^{-6}q;q^3)_\infty\right)
\end{aligned}$$

by theorem 3. Applying the chain rule to

$$(x^6q^2;q^3)_\infty(x^{-6}q;q^3)_\infty = \prod_{k=0}^{\infty}(1-x^6q^{3k+2})(1-x^{-6}q^{3k+1})$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^6q^2;q^3)_\infty(x^{-6}q;q^3)_\infty = (x^6q^2;q^3)_\infty(x^{-6}q;q^3)_\infty\sum_{k=0}^{\infty}\frac{(-6x^5q^{3k+2})}{(1-x^6q^{3k+2})}+\frac{(6x^{-7}q^{3k+1})}{(1-x^{-6}q^{3k+1})}$$

so

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)x^{6n}q^{n(3n+1)/2} &= (q^3)_\infty(x^6q^2;q^3)_\infty(x^{-6}q;q^3)_\infty \times \\
&\quad \left(1+x\sum_{k=0}^{\infty}\frac{(-6x^5q^{3k+2})}{(1-x^6q^{3k+2})}+\frac{(6x^{-7}q^{3k+1})}{(1-x^{-6}q^{3k+1})}\right)
\end{aligned}$$

If we now let $x=1$ we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)q^{n(3n+1)/2} &= (q^3)_\infty(q^2;q^3)_\infty(q;q^3)_\infty \times \\
&\quad \left(1+6\sum_{k=0}^{\infty}\frac{(-q^{3k+2})}{(1-q^{3k+2})}+\frac{(q^{3k+1})}{(1-q^{3k+1})}\right)
\end{aligned}$$

And since $(q^3)_\infty(q^2;q^3)_\infty(q;q^3)_\infty\times = (q)_\infty$ we are done. □

Corollary 1. *The number of Eisenstein integers with norm $r^2 = x^2+xy+y^2$ is 1 if $r^2=0$; and otherwise is 6 times (the number of divisors of r^2 of the form $3n+1$ minus the number of divisors of r^2 of the form $3n+2$).*

Proof: we can read this straight off the generating function of theorem 6. The sum on the right hand size contributes nothing to q^0, so the coefficient of q^0 is just 1 from the constant outside the sum. Recalling the geometric progression $\frac{x}{1-x} = x + x^2 + x^3 + \cdots$, every number of the form $3k + 1$ contributes 6 to each of its products, and every number of the form $3k + 2$ subtracts 6 from each of its products. $\square$

Counting Eisenstein integers with a bounded norm

Theorem 7. *If $f(r)$ denotes the number of Eisenstein integers (lattice points) with norm no greater than r^2 (within a circle of radius r from the origin), then*

$$f(r) = 1 + 6\sum_{k=0}^{\infty} \left\lfloor \frac{r^2}{3k+1} \right\rfloor - \left\lfloor \frac{r^2}{3k+2} \right\rfloor$$

Proof: sum over the corollary for $0 \leq m^2 + mn + n^2 \leq r^2$. The constant 1 arises from radius 0; then each number of the form $3k + 1$ contributes 6 for each of its products in the range $[1, r^2]$, and each number of the form $3k+2$ subtracts 6 for each of its products in the range $[1, r^2]$. $\square$

References

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